Irreducible finite-dimensional representations of equivariant map algebras

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December 2009

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Full details: arXiv:0906.5189

Outline

Goal: Classify the irreducible finite-dimensional representations of a new class of Lie algebras.

Overview:

1. Equivariant map algebras
2. Evaluation maps
3. Irreducible finite-dimensional representations
4. Applications/Examples
    - recover some known classifications
    - produce some new classifications

Notation

- k algebraically closed field of characteristic 0
- A commutative associative k-algebra, finitely generated and without nilpotents
- X associated affine algebraic variety ($A = \mathcal{O}_X(X)$)
- $\mathfrak{g}$ finite-dimensional (Lie) algebra over k
- Γ finite group acting on A by automorphisms of A ($\leftrightarrow$ acting on X by automorphisms of X) and on $\mathfrak{g}$ by automorphisms of $\mathfrak{g}$

(Generalizations:

- A commutative associative k-algebra
- X associated affine scheme, or vice versa
- replace $x \in X$ by "k-rational points")

Untwisted Map algebras

Recall X affine algebraic variety, A coordinate ring, $\mathfrak{g}$ (Lie) algebra

Definition

$M(X, \mathfrak{g}) = \mathfrak{g} \otimes A \cong \{\alpha : X \to \mathfrak{g} : \alpha \text{ regular map}\}$

(Lie) algebra :

$$\begin{aligned} [u \otimes a, v \otimes b] &= [u, v]_{\mathfrak{g}} \otimes ab \qquad (u, v \in \mathfrak{g}, a, b \in A) \\ [\alpha, \beta](x) &= [\alpha(x), \beta(x)]_{\mathfrak{g}} \qquad (\alpha, \beta : X \to \mathfrak{g}, x \in X) \end{aligned}$$

Recall $M(X, \mathfrak{g}) = \mathfrak{g} \otimes A = \{\alpha : X \to \mathfrak{g} : \alpha \text{ regular}\}$

Examples:

- $A = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, $X = (k^\times)^n$, $k^\times = k \setminus \{0\}$
  $n = 1$: $M(X, \mathfrak{g})$ loop algebra
- $A = k[t_1, \dots, t_n]$, $X = k^n$,
  $M(X, \mathfrak{g})$ current algebra
- $A = k \times \cdots \times k$ (n factors), $X = \{x_1, \dots, x_n\}$,
  $M(X, \mathfrak{g}) \cong \mathfrak{g} \oplus \cdots \oplus \mathfrak{g} = \mathfrak{g}^{\oplus n}$ (n terms)

Equivariant map algebras (EMAs)

Γ finite group, $A \curvearrowleft \Gamma \curvearrowright \mathfrak{g}$, hence $\Gamma \curvearrowright M(X,\mathfrak{g}) = \mathfrak{g} \otimes A =$ regular maps $X \to \mathfrak{g}$:

$\gamma \cdot (u \otimes a) = (\gamma \cdot u) \otimes (\gamma \cdot a)$, or $(\gamma \cdot \alpha)(x) = \gamma \cdot \alpha(\gamma^{-1} \cdot x)$

Definition (equivariant map algebras)

$$\begin{aligned} M(X,\mathfrak{g})^\Gamma &= (\mathfrak{g} \otimes A)^\Gamma \\ &= \{\alpha : X \to \mathfrak{g} : \alpha \text{ regular}, \alpha(\gamma \cdot x) = \gamma \cdot \alpha(x)\ \forall \gamma \in \Gamma, x \in X\} \end{aligned}$$

$M(X,\mathfrak{g})^\Gamma < M(X,\mathfrak{g})$

Note: $(\mathfrak{g} \otimes A)^\Gamma \supset \mathfrak{g}^\Gamma \otimes A^\Gamma$

Examples: Trivial Γ-actions

- Γ acts trivially on A, i.e., on X: $M(X,\mathfrak{g})^\Gamma = M(X,\mathfrak{g}^\Gamma)$ (untwisted EMA)
- Γ acts trivially on $\mathfrak{g}$: $M(X,\mathfrak{g})^\Gamma = \mathfrak{g}\otimes A^\Gamma = M(X/\!/\Gamma,\mathfrak{g})$
  $X/\!/\Gamma = \mathrm{Spec}(A^\Gamma)$ geometric quotient

Example: Multiloop algebras

$$\Gamma = \text{abelian}, \quad X = (k^\times)^n, \quad \mathfrak{g} = \text{any algebra}$$

Γ abelian: $\Gamma = \mathbb{Z}/m_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m_n\mathbb{Z} \ni \bar{p} = (\bar{p}_1, \dots, \bar{p}_n)$

$A = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, Γ-action: $\bar{p} \cdot t_i = \zeta_i^{-p_i} t_i$, ζ_i primitive m_ith root of 1,

$X = (k^\times)^n$, Γ-action: $\bar{p} \cdot (x_1, \dots, x_n) = (\zeta_1^{p_1} x_1, \dots, \zeta_n^{p_n} x_n)$,

Γ action on $\mathfrak{g} \leftrightarrow \sigma_1, \dots, \sigma_n \in \mathrm{Aut}(\mathfrak{g})$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $i \neq j$, $\sigma_i^{m_i} = \mathrm{Id}_{\mathfrak{g}}$,
Γ-grading of $\mathfrak{g}$ by joint eigenspaces $\mathfrak{g}_{\bar{p}}$: $\mathfrak{g} = \bigoplus_{\bar{p} \in \Gamma} \mathfrak{g}_{\bar{p}}$

$$M(X, \mathfrak{g})^\Gamma = \bigoplus_{p \in \mathbb{Z}^n} \mathfrak{g}_{\bar{p}} \otimes kt^p \quad \textit{twisted multiloop algebra}$$

$n = 1$, $\mathfrak{g}$ simple Lie: $\rightsquigarrow$ affine Kac-Moody Lie algebras

n arbitrary, $\mathfrak{g}$ simple Lie: $\rightsquigarrow$ toroidal Lie algebras

Example: generalized Onsager algebra

$$\Gamma = \{1, \sigma\}, \quad A = k[t^{\pm 1}], \quad X = k^{\times}, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- Γ acts on A by $\sigma \cdot t = t^{-1}$, on X by $\sigma \cdot x = x^{-1}$
- Γ acts on $\mathfrak{g}$ by any involution

When Γ acts on Lie algebra $\mathfrak{g}$ by the Chevalley involution, we write

$$\mathcal{O}(\mathfrak{g}) = M(X, \mathfrak{g})^{\Gamma}$$

Remarks

- If $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_2)$ is isomorphic to the Onsager algebra (Roan 1991)
  - Key ingredient in Onsager's original solution of the 2D Ising model
- For $k = \mathbb{C}$, $\mathcal{O}(\mathfrak{sl}_n)$ was studied by Uglov and Ivanov (1996)

Evaluation maps

$\mathfrak{M} = M(X, \mathfrak{g})^\Gamma = (\mathfrak{g} \otimes A)^\Gamma$

$x \in X$, $\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}$

$\mathfrak{g}^x = \{u \in \mathfrak{g} : \gamma \cdot u = u, \ \forall \gamma \in \Gamma_x\} < \mathfrak{g}$

Important: $\alpha \in \mathfrak{M} \implies \alpha(x) \in \mathfrak{g}^x$

since $\alpha(x) = \alpha(\gamma \cdot x) = \gamma \cdot \alpha(x)$ for all $\gamma \in \Gamma_x$.

Definition (Evaluation at $x \in X$)

$$\mathrm{ev}_x : \mathfrak{M} \to \mathfrak{g}^x, \quad \alpha \mapsto \alpha(x)$$

(For $\mathfrak{M} = (\mathfrak{g} \otimes A)^\Gamma$: $x \leftrightarrow \mathfrak{m}$ maximal ideal of A, $A \to A/\mathfrak{m} \cong k$, $a \mapsto \bar{a}$
then $\mathrm{ev}_x(\sum_i u_i \otimes a_i) = \sum_i u_i \bar{a}_i$)

Ideals in $\mathfrak{M}$ ($\mathfrak{g}$ arbitrary algebra)

Proposition

Assume $\mathfrak{K} \lhd \mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ and $\mathfrak{M}/\mathfrak{K}$ simple and finite-dimensional. Then exists $x \in X$ such that

$$\begin{tikzcd} \mathfrak{M} \arrow[rr, two heads, "\mathrm{can}"] \arrow[dr, two heads, "\mathrm{ev}_x"'] & & \mathfrak{M}/\mathfrak{K} \\ & \mathfrak{g}^x \arrow[ur, two heads] & \end{tikzcd}$$

Structure of all $\mathfrak{K}$ and $\mathfrak{M}/\mathfrak{K}$:

Lau 2008 (multiloop), Elduque 2007 (Onsager)

Associative algebras

Corollary

$\mathfrak{g}$ *associative,* $\rho : \mathfrak{M} \to \mathrm{End}_K(V)$ *finite-dimensional irreducible representation. Then there exists* $x \in X$ *and a* $\rho_x : \mathfrak{g}^x \to \mathrm{End}_k(V)$ *such that* $\rho = \rho_x \circ \mathrm{ev}_x$ *:*

$$\mathfrak{M} \xrightarrow{\mathrm{ev}_x} \mathfrak{g}^x \xrightarrow{\rho_x} \mathrm{End}_k(V)$$

Small representations

From now on: $\mathfrak{g}$ is a Lie algebra, hence so is $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$.

Small representation = irreducible finite-dimensional representation

Example: Very small representations

Every 1-dimensional representation of a Lie algebra L is small.

Construction:

To $\lambda \in L^*, \lambda([L, L]) = 0$ we associate the rep ρ_λ, $\rho_\lambda(l)(s) = \lambda(l)s$ for $s \in k$.

$$\begin{aligned} (L/[L, L])^* \quad &\cong \quad \{\text{1-dim rep}\}, \quad \lambda \mapsto \rho_\lambda \\ \lambda = \mu \quad &\iff \quad \rho_\lambda \cong \rho_\mu \end{aligned}$$

Evaluations representations

Recall: small = irreducible finite-dimensional

Definition (Evaluation representation)

A (small) evaluation representation of $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ is a representation ρ for which there exist

1. $\mathbf{x} \subset X$ finite, $\Gamma \cdot x_1 \cap \Gamma \cdot x_2 = \emptyset$ if $x_i \in \mathbf{x}$, $x_1 \neq x_2$,
2. $\rho_x : \mathfrak{g}^x \mapsto \mathfrak{gl}(V_x)$ (small) representation for every $x \in \mathbf{x}$

such that $\rho = (\otimes_{x \in \mathbf{x}} \rho_x) \circ \mathrm{ev}_\mathbf{x}$:

$$\mathfrak{M} \xrightarrow{\mathrm{ev}_\mathbf{x}} \oplus_{x \in \mathbf{x}} \mathfrak{g}^x \xrightarrow{\otimes \rho_x} \otimes_{x \in \mathbf{x}} \mathfrak{gl}_k(V_x).$$

where $\mathrm{ev}_\mathbf{x}(\alpha) = (\alpha(x))_{x \in \mathbf{x}}$

Lemma: A small evaluation rep is a small rep

NEW

- To $x \in \mathbf{x}$ we associate a rep of $\mathfrak{g}^x$ instead of $\mathfrak{g}$. A rep of $\mathfrak{g}^x$ does in general not extend to a rep of $\mathfrak{g}$, so we get more reps.
- If all $\mathfrak{g}^x = \mathfrak{g}$, e.g. Γ acts freely on X (multiloop case), then we get the usual definition.
- The small reps ρ_x need not be faithful

WHY the more general concept?
We will get a more uniform description!

Recall small= finite-dimensional irreducible

Theorem (N.-Savage-Senesi)

The small representations of $\mathfrak{M}$ are

$$\textit{(1-dim. rep)} \otimes \textit{(small evaluation rep)}$$

Proof: (1) The small rep of any Lie algebra L are (1-dim. rep) $\otimes$ ρ_s where ρ_s has $L/\operatorname{Ker}(\rho_s)$ finite-dimensional semisimple.

(2) Prop. implies for $L = \mathfrak{M}$: ρ_s is evaluation rep

Isomorphism classes

- $\mathcal{S}$ = isomorphism classes of small reps
- $\mathcal{R}_x$ = isomorphism classes of small reps of $\mathfrak{g}^x$, $x \in X$,
- $\mathcal{R} = \bigsqcup_{x \in X} \mathcal{R}_x$, Γ acts on $\mathcal{R}$ by $\gamma \cdot [\rho_x] = [\rho_x \circ \gamma^{-1}] \in \mathcal{R}_{\gamma \cdot x}$ for $[\rho_x] \in \mathcal{R}_x$
- $\mathcal{F}$ = equivariant functions $\psi : X \to \mathcal{R}$, $\psi(x) \in \mathcal{R}_x$, $\psi(x) = 0$ for almost all $x \in X$

Define

$$\mathcal{F} \to \mathcal{S}, \quad [\psi] \to [\mathrm{ev}_\psi]$$

where to ψ we associate

- $\mathbf{x} \subset \mathcal{F}$ by $\mathrm{supp}(\psi) = \{x \in X : \psi(x) \neq 0\} = \Gamma \cdot \mathbf{x}$, $\Gamma \cdot x_1 \cap \Gamma \cdot x_2 = \emptyset$ if $x_i \in X$, $x_1 \neq x_2$,
- then choose $\psi(x) = [\rho_x]$ for $x \in \mathbf{x}$, and define

$$\mathrm{ev}_\psi = \left(\bigotimes_{x \in \mathbf{x}} \rho_x \right) \circ \mathrm{ev}_{\mathbf{x}}$$

Theorem (N-Savage-Senesi)

For $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$*:*

$$\big(\mathfrak{M}/[\mathfrak{M},\mathfrak{M}]\big)^* \times \mathcal{F} \to \mathcal{S}, \quad (\lambda, [\psi]) \to [\rho_\lambda \otimes \mathrm{ev}_\psi]$$

is surjective, and injective on each factor. In particular:

$$\mathfrak{M} = [\mathfrak{M},\mathfrak{M}] \implies \mathcal{F} \cong \mathcal{S}$$

When are all small reps evaluation reps, i.e., $\mathcal{F} \cong \mathcal{S}$?

Recall: small $= (\mathfrak{M}/[\mathfrak{M},\mathfrak{M}])^* \otimes$ eval rep.

Sufficient condition: $\mathfrak{M} = [\mathfrak{M},\mathfrak{M}]$.

Example (Γ trivial, $\mathfrak{g}$ perfect)

(since then $\mathfrak{M} = \mathfrak{g} \otimes A$ and $[\mathfrak{M},\mathfrak{M}] = [\mathfrak{g},\mathfrak{g}] \otimes A$)

Special case: $\mathfrak{g}$ semisimple

Chari 1986, $A = k[t^{\pm 1}]$ (untwisted loop algebra)

Feigin-Loktev 2004, A arbitrary

Chari-Fourier-Khandi, A arbitrary (preprint 2009)

Example (Multiloop algebra)

Recall: Γ *abelian*, $A = k[t_1^{\pm 1}, \dots, t_n^{\pm 1}], X = (k^\times)^n, \mathfrak{g}$ perfect,

$$\mathfrak{M} = \bigoplus_{p \in \mathbb{Z}^n} \mathfrak{g}_{\bar{p}} \otimes kt^p.$$

Fact (Allison-Berman-Pianzola): $\mathfrak{M}$ is perfect.

Corollary

If $\mathfrak{M}$ is a (twisted) multiloop algebra, then all small representations are evaluation representations: $\mathcal{F} \cong \mathcal{S}$.

Remarks

1. recovers results of Chari-Pressley (for loop algebras), Rao, and Batra, Lau (multiloop algebras).
2. Action of Γ on X is free, so $\mathfrak{g}^x = \mathfrak{g}$ for all $x \in X$, so the more general notion of evaluation rep does not play a role.

When are all small reps evaluation reps ($\mathcal{F} \cong \mathcal{S}$)?

Obvious answer:

Since small reps are (1-dim rep) $\otimes$ (eval rep) it suffices to know if there are one-dimensional reps which are not evaluation reps.

The critical points are

Definition

$$\tilde{X} = \{x \in X : \mathfrak{g}^x \neq [\mathfrak{g}^x, \mathfrak{g}^x]\}$$

Recall: $\mathfrak{g}^x = [\mathfrak{g}^x, \mathfrak{g}^x] \iff$ all one-dimensional reps of $\mathfrak{g}^x$ are trivial. Thus:

$\tilde{X}$ is precisely the set of points where we can place nontrivial one-dimensional evaluation representations.

New question

When can non-trivial 1-dimensional representations be realized as evaluation representations?

Use our more general definition of an evaluation representation!

Let $\mathbf{x}$ be a finite subset of X, consider the commutative diagram:

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, two heads, "\mathrm{ev}_{\mathbf{x}}"] \arrow[dr, two heads] & \oplus_{x \in \mathbf{x}} \mathfrak{g}^x \arrow[r, two heads] & \oplus_{x \in \mathbf{x}} \mathfrak{g}^x / [\mathfrak{g}^x, \mathfrak{g}^x] \\
& \mathfrak{M}/[\mathfrak{M}, \mathfrak{M}] \arrow[ur, two heads, "f"'] &
\end{tikzcd}$$

Proposition

If $|\tilde{X}| < \infty$, choose $\mathbf{x}$ such that $\tilde{X} = \Gamma \cdot \mathbf{x}$. Then all small reps are eval reps if and only if $\ker f = 0$. This is true if and only if

$$[\mathfrak{M}, \mathfrak{M}] = \{\alpha \in \mathfrak{M} \mid \alpha(x) = [\mathfrak{g}^x, \mathfrak{g}^x] \;\forall\; x \in X\}.$$

Application: generalized Onsager algebra again

Recall

$$\Gamma = \{1, \sigma\}, \quad X = k^\times, \quad \mathfrak{g} = \text{simple Lie algebra}$$

- Γ acts on X by $\sigma \cdot x = x^{-1}$
- Γ acts on $\mathfrak{g}$ by any involution

Corollary

For Γ, X, $\mathfrak{g}$ as above, all small reps are evaluation reps.

Benkart-Lau

- There are two types of points of $X = k^\times$:
  - $x \in \{\pm 1\} \implies \Gamma_x = \Gamma = \mathbb{Z}_2$, $\mathfrak{g}^x = \mathfrak{g}^\Gamma$
  - $x \notin \{\pm 1\} \implies \Gamma_x = \{1\}$, $\mathfrak{g}^x = \mathfrak{g}$
- $\mathfrak{g}^\Gamma$ can be semisimple or have a one-dimensional center
- When $\mathfrak{g}^\Gamma$ is semisimple, $\mathfrak{M}$ is perfect: Done!
- When $\mathfrak{g}^\Gamma$ has a one-dimensional center: we can place (nontrivial) one-dim reps of $\mathfrak{g}^\Gamma$ at the points ± 1
- the generalized Onsager algebra is not perfect
- but the map f of the Prop. is bijective! Done!
- Note: under our more general definition of evaluation rep, all small reps are evaluation reps
- under classical notion of evaluation rep, there are small reps which are not evaluation reps

Moral: The more general definition of evaluation rep allows for a more uniform classification.

Special case: Onsager algebra

- When $k = \mathbb{C}$ and Γ acts on $\mathfrak{g} = \mathfrak{sl}_2$ by the Chevalley involution, then

$$\mathcal{O}(\mathfrak{sl}_2) \stackrel{\text{def}}{=} M(X, \mathfrak{sl}_2)^\Gamma$$

  is the Onsager algebra
- $\mathfrak{g}^{\{\pm 1\}}$ is one-dimensional abelian and $\mathcal{O}(\mathfrak{sl}_2)$ is not perfect.
- Small reps of $\mathcal{O}(\mathfrak{sl}_2)$ were classified previously (Date-Roan 2000)
  - classical definition of evaluation rep was used
  - not all small reps were evaluation reps
  - this necessitated the introduction of the type of a representation

Note: For the other cases, the classification seems to be new.

When are all small reps evaluation reps?

Case $\tilde{X} = \{x \in X : \mathfrak{g}^x \neq [\mathfrak{g}^x, \mathfrak{g}^x]\}$ finite.

Recall Prop: Small reps = eval reps iff
$[\mathfrak{M}, \mathfrak{M}] = \mathfrak{M}^d = \{\alpha \in \mathfrak{M} : \alpha(x) \in [\mathfrak{g}^x, \mathfrak{g}^x] \ \forall x \in \tilde{X}\}$

Example ($|\tilde{X}| < \infty$, not all small reps are evaluation reps)

$$\Gamma = \{1, \sigma\}, \quad \mathfrak{g} = \mathfrak{sl}_2(k), \quad X = \text{cuspidal cubic curve}$$

- Γ acts on $\mathfrak{g}$ by Chevalley involution
- $X = Z(x^2 - y^3) = \{(x, y) \mid x^2 = y^3\} \subset k^2$, $A = k[x, y]/(x^2 - y^3)$,
- Γ acts on k^2 by $\sigma \cdot (x, y) = (-x, y)$, fixes $x^2 - y^3$, hence induces an action of Γ on X.
- Only fixed point is the origin.
- Thus $\tilde{X} = \{0\}$ and so $|\tilde{X}| < \infty$
- But $\mathfrak{M}^d/[\mathfrak{M}, \mathfrak{M}] \cong yk[y]/(y^3) \neq 0$

Thus $[\mathfrak{M}, \mathfrak{M}] \subsetneq \mathfrak{M}^d$ and so $\mathfrak{M}$ has small reps that are not eval reps.

Note: X has a singularity at 0.

When are all small rep evaluation reps?

Case $\tilde{X} = \{x \in X : \mathfrak{g}^x \neq [\mathfrak{g}^x, \mathfrak{g}^x]\}$ infinite

Proposition (N-Savage-Senesi 2009)

If $|\tilde{X}| = \infty$, then $\mathfrak{M} = M(X, \mathfrak{g})^\Gamma$ has one-dimensional representations that are not evaluation representations.

(For X a scheme suppose X Noetherian)

Example ($\tilde{X}$ infinite)

$$\Gamma = \{1, \sigma\}, \quad A = k[t_1, t_2], \quad X = k^2, \quad \mathfrak{g} = \mathfrak{sl}_2(k)$$

- $\sigma \cdot (x_1, x_2) = (x_1, -x_2)$, $(x_1, x_2) \in k^2$
- σ acts as Chevalley involution on $\mathfrak{g}$

Then

- $x = (x_1, x_2) \in k^2,\ x_2 \neq 0 \implies \Gamma_x = \{1\} \implies \mathfrak{g}^x = \mathfrak{g}$
- $x = (x_1, 0) \in k^2 \implies \Gamma_x = \Gamma \implies \mathfrak{g}^x \cong k$ (abelian)

Thus

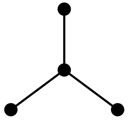
$$\tilde{X} = \{(x_1, 0) \mid x_1 \in k\} \quad \text{and so} \quad |\tilde{X}| = \infty.$$

Therefore $\mathfrak{M}$ has one-dimensional reps that are not eval reps.

A nonabelian example (Γ nonabelian)

Example (Γ nonabelian)

$\Gamma = S_3, \quad X = \mathbb{P}^1 \backslash \{0, 1, \infty\}, \quad A = k[t^{\pm 1}, (t-1)^{-1}], \quad \mathfrak{g} = \mathfrak{so}_8 \quad (\text{type } \mathrm{D}_4)$



D_4

- symmetry group of Dynkin diagram of $\mathfrak{g}$ is S_3
- Γ acts on $\mathfrak{g}$ by diagram automorphisms
- for any permutation of the set $\{0, 1, \infty\}$, $\exists!$ homography of $\mathbb{P}^1$ inducing that permutation
- so Γ acts naturally on X

The corresponding equivariant map algebra $M(X, \mathfrak{g})^\Gamma$ is perfect. Hence $\mathcal{F} \cong \mathcal{S}$

A nonabelian example

Points with nontrivial stabilizer:

x	Γ_x	Type of $\mathfrak{g}^x$
-1	$\{\mathrm{Id}, (0\,\infty)\} \cong \mathbb{Z}_2$	B_3
2	$\{\mathrm{Id}, (1\,\infty)\} \cong \mathbb{Z}_2$	B_3
$\frac{1}{2}$	$\{\mathrm{Id}, (0\,1)\} \cong \mathbb{Z}_2$	B_3
$e^{\pm\pi i/3}$	$\{\mathrm{Id}, (0\,1\,\infty), (0\,\infty\,1)\} \cong \mathbb{Z}_3$	G_2

The sets

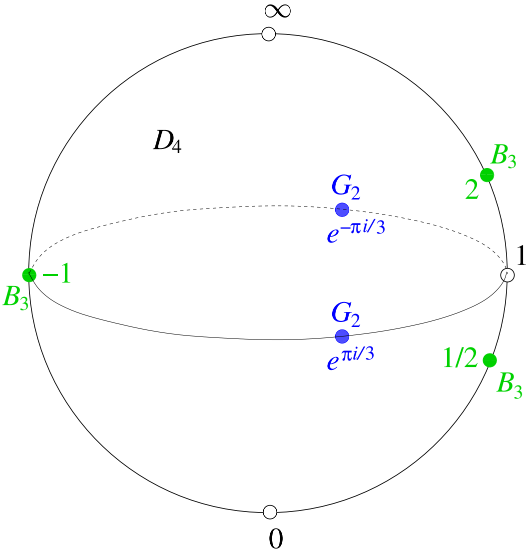
$$\left\{-1, 2, \frac{1}{2}\right\} \quad \text{and} \quad \left\{e^{\pi i/3}, e^{-\pi i/3}\right\}$$

are Γ-orbits.

So elements of $\mathcal{F}$ can assign

- irreps of type B_3 to the 3-element orbit
- irreps of type G_2 to the 2-element orbit
- irreps of type D_4 to the other points (6-element orbits)

A nonabelian example



Conclusion

- Equivariant map algebras provide a framework for many interesting examples
- Interplay between algebra and geometry
- Many open problems